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MINIMAL TRANSFORMATIONS WITH NO COMMON FACTOR NEED NOT BE DISJOINT

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ABSTRACT
A countable family of minimal transformations $(X, \mathbf{Z})$ is described for which no pair have a non-trivial common factor, and so that no pair is disjoint. This answers in the negative a question of H. Furstenberg.

§1. If (X_i, T) are minimal actions of a group T then (X_2, T) is a *factor* (X_1, T) if there is a T equivariant map π from X_1 onto X_2. A pair of minimal actions (X_i, T), $i = 1,2$, are said to be *disjoint* if whenever they are both factors of a minimal action (X, T) via $\pi_i : X \to X_i$, $i = 1,2$ the maps factor through some surjective map $\pi : X \to X_1 \times X_2$. An equivalent condition is that the product action $(X_1 \times X_2, T)$ is minimal. It is easy to see that if (X_1, T) and (X_2, T) have a non-trivial common factor then they cannot be disjoint. In [3], H. Furstenberg introduced the concept of disjointness for $T = \mathbf{Z}$ and asked if the converse holds, i.e. does disjointness follow from the non-existence of a common factor. Already in [8], A. Knapp pointed out that the converse is false for quite simple non-commutative groups T. For abelian groups T several results in the positive direction were obtained, cf. [2]. For the analogous question concerning ergodic measure preserving transformations D. Rudolph and J. P. Thouvenot constructed in [13] an example showing that the converse is false, that is to say, not having a common factor need not imply disjointness.

In this paper we point out the existence of a countable family of minimal real flows $(X_i, \{h_t\}_{t\in\mathbf{R}})$ for which no pair have a common factor and so that no pair is disjoint. Moreover, the family of "time one" transformations of these flows (X_i, h_1) (which are also minimal) has the same properties, namely no pair have a common factor and no pair is disjoint.

Our flows are the classical horocycle flows on different compact surfaces of constant negative curvature; we make essential use of the recent deep studies of

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M. Ratner concerning the structure of these flows, [9, 10]. We owe a great debt to D. Kazhdan for having shown us how to construct a family of uniform subgroups of $\mathrm{SL}(2,\mathbf{R})$ that has the properties that we needed. For the remainder of the paper G will denote $\mathrm{SL}(2,\mathbf{R})$ and h_t the horocycle subgroup acting on G/Γ where Γ is a uniform (i.e. discrete and co-compact) subgroup of G. We will need three results concerning the horocycle flows:

THEOREM A (H. Furstenberg [4]). *The horocycle flow h_t on a compact surface G/Γ is uniquely ergodic, i.e. it has a unique invariant measure.*

THEOREM B (M. Ratner [9]). *If for two horocycle flows, $(G/\Gamma_1, h_t)$ and $(G/\Gamma_2, h_t)$, the measure preserving transformations $(G/\Gamma_1, h_1)$ and $(G/\Gamma_2, h_1)$ are isomorphic, then Γ_1 and Γ_2 are conjugate subgroups of G.*

THEOREM C (M. Ratner [10]). *If the measure preserving transformation (X, S) is a measure theoretic factor of a horocycle "time one" transformation $(G/\Gamma, h_1)$ then (X, S) is measure theoretically isomorphic to a horocycle transformation $(G/\Gamma_1, h_1)$ with $\Gamma_1 \supset \Gamma$.*

The minimality of the horocycle flow is a well known classical result. We remark that since all the h_t are conjugate to either h_1 or h_{-1} (by the geodesic flow), it follows from Theorem A that for each t and in particular for $t = 1$, $(G/\Gamma, h_t)$ is uniquely ergodic and minimal.

Our family $(X_i, \mathbf{R})$ will be $(G/\Gamma_i, h_t)$ where $\{\Gamma_i\}$ is a sequence of uniform subgroups satisfying certain conditions. The next theorem asserts the existence of the required family.

THEOREM 1. *There exists a countable family of uniform subgroups $\{\Gamma_i\}$ of G that satisfy:*

(i) *for each i, j, $\Gamma_i \cap \Gamma_j$ is of finite index in both Γ_i and Γ_j;*

(ii) *for all $i \neq j$ and $g \in G$, Γ_i and $g\Gamma_j g^{-1}$ generate a non-discrete subgroup of G.*

The construction of such a family will be carried out in §2. We proceed to show that the $(G/\Gamma_i, h_t)$ have the properties announced above. We will discuss the family $(G/\Gamma_i, h_1)$; the argument for the family of real flows $(G/\Gamma_i, h_t)$ is analogous. To begin with, by (i), both $(G/\Gamma_i, h_1)$ and $(G/\Gamma_j, h_1)$ are factors of the horocycle flow $(G/\Gamma_i \cap \Gamma_j, h_1)$ with finite fibers so that they certainly are not disjoint. Suppose now that (X, S) is a common factor. By Theorem A, (X, S) is a factor of a uniquely ergodic system and hence is uniquely ergodic, say with invariant measure μ. The uniqueness implies that (X, S, μ) is a measure theoretic factor of both $(G/\Gamma_i, h_1)$ and $(G/\Gamma_j, h_1)$. Thus by Theorem C, (X, S, μ) is isomorphic to both $(G/\hat{\Gamma}_i, h_1)$ and $(G/\hat{\Gamma}_j, h_1)$ with $\hat{\Gamma}_i$, $\hat{\Gamma}_j$ uniform subgroups satisfying $\hat{\Gamma}_i \supset \Gamma_i$ and $\hat{\Gamma}_j \supset \Gamma_j$. Now Theorem B implies that there is some $g \in G$ with $g\hat{\Gamma}_j g^{-1} = \hat{\Gamma}_i$ and thus both Γ_i and $g\Gamma_j g^{-1}$ lie in the same uniform subgroup $\hat{\Gamma}_i$ which for $i \neq j$ contradicts property (ii). We have established the following result:

THEOREM 2. *If the uniform subgroups* Γ_i *satisfy the conclusion of Theorem* 1(i) *and* (ii), *then the minimal transformations* $(G/\Gamma_i, h_1)$ *are pairwise non-disjoint and pairwise have no common factor.*

§2. To construct the Γ_i's begin with a quaternion subgroup Γ. To be definite set

$$\Gamma = \left\{ \begin{pmatrix} a + b\sqrt{2} & c + d\sqrt{2} \\ 3(c - d\sqrt{2}) & a - b\sqrt{2} \end{pmatrix} : a, b, c, d \in \mathbf{Z},\ a^2 - 2b^2 - 3c^2 + 6d^2 = 1 \right\}.$$

The group $\Gamma \subset G$ and is co-compact ([5]). We let

$$D_{\mathbf{Q}} = \left\{ \begin{pmatrix} a + b\sqrt{2} & c + d\sqrt{2} \\ 3(c - d\sqrt{2}) & a - b\sqrt{2} \end{pmatrix} : a, b, c, d \in \mathbf{Q} \right\}$$

and recall that $D_{\mathbf{Q}}$ is a division algebra. At this point we need a lemma which can be proved using the rudiments of the Hasse–Minkowski theory, as described in [1, ch. 1], for example. Since the result is fairly routine we give only an outline of the proof.

LEMMA. *For any prime* $p, p \equiv 1 \pmod{24}$ *the quadratic form*

$$px^2 + 2y^2 + 3z^2 - 6w^2 = 0$$

has a non-trivial solution in integers x, y, z, w.

PROOF. According to the Hasse–Minkowski theorem we need only check that the form represents zero over the reals and over the q-adic numbers for all prime q. For the real field this is clear, and for any $q \neq 2, 3$ the form has at least three coefficients which are q-adic units so that once again the general theory gives that it represents zero over the q-adics for $q \neq 2, 3$. For $q = 2, 3$ one checks directly that zero is represented; here one uses the condition $p \equiv 1 \pmod{24}$. □

By the lemma we have rational numbers r, s, t such that $2r^2 + 3s^2 - 6t^2 = -p$ and thus setting

$$\gamma_p = \begin{pmatrix} r\sqrt{2} & s + t\sqrt{2} \\ 3(s - t\sqrt{2}) & -r\sqrt{2} \end{pmatrix}$$

we have $\gamma_p \in D_Q$, and $\gamma_p^2 = -pI$, where I is the identity matrix. Denoting as usual the conjugation with γ_p of Γ by Γ^{γ_p} we set $\Gamma(p) = \Gamma \cap \Gamma^{\gamma_p}$ and $\Gamma_p = \{\Gamma(p), \gamma_p/\sqrt{p}\}$ the group generated by $\Gamma(p)$ and $\gamma_p/\sqrt{p}$. Clearly $\Gamma_p \subset G$.

One can write γ_p in the form $(1/a)A$ where $a \in \mathbf{Z}$ and A has integral entries. Let $\Lambda \subset \Gamma$ consist of the matrices congruent to I (mod pa^2), then Λ is a subgroup of finite index in Γ. On the other hand, $\Lambda^{\gamma_p} \subset \Gamma$ and hence $\Lambda^{\gamma_p} \subset \Gamma \cap \Gamma^{\gamma_p}$. Conjugating with γ_p and recalling that γ_p^2 is a scalar, we obtain $\Lambda \subset \Gamma \cap \Gamma^{\gamma_p}$. It follows that $\Gamma(p) = \Gamma \cap \Gamma^{\gamma_p}$ has finite index in Γ. Since $\Gamma(p)$ is of index 2 in Γ_p it follows that Γ_p is uniform and the family $\{\Gamma_\iota\}$ of Theorem 1 is simply $\{\Gamma_p\}_{p \in P}$ where

$$P = \{\text{primes } p : p \equiv 1 \pmod{24}\}.$$

Conclusion (i) of Theorem 1 for $p, q \in P$ follows upon consideration of the subgroup of matrices of Γ congruent to I (mod m) for a suitable m as above. The remainder of the section is devoted to proving (ii).

Fix two distinct elements p, q in P and $g \in G$ and let $\Delta = \{\Gamma_p, \Gamma_q^g\}$, the subgroup generated by Γ_p and Γ_q^g. We suppose that Δ is discrete and aim at deducing a contradiction. Since Γ_p and Γ_q^g are uniform, each is of finite index in Δ and thus so is their intersection. Let $\Lambda_0 = \Gamma(p) \cap \Gamma(q)$ and verify that $\Lambda_1 = \Lambda_0 \cap \Lambda_0^g$ is of finite index in Γ. Since $\Lambda_1^{g^{-1}} \subset \Gamma$, by considering the algebra generated by Λ_1 over $\mathbf{Q}$ we conclude that $D_{\mathbf{Q}}^g = D_{\mathbf{Q}}$. In particular $\gamma_q^g = \delta$ is an element of $D_{\mathbf{Q}}$ and by construction its determinant is q. All that we shall need for the continuation is the existence of a $\delta \in D_{\mathbf{Q}}$ with determinant $= q$, such that $\delta/\sqrt{q}$ together with Γ_p generates a group which is a finite extension of $\Gamma(p)$.

At this point we introduce the q-adic completion of $\mathbf{Q}$, $\mathbf{Q}_q$ and let $D_{\mathbf{Q}_q} = D_{\mathbf{Q}} \otimes \mathbf{Q}_q$. The latter is isomorphic to $M(2, \mathbf{Q}_q)$ since $\sqrt{2} \in \mathbf{Q}_q$ by quadratic reciprocity. There is a natural mapping of $\mathrm{GL}(2, \mathbf{Q}_q)$, which is the multiplicative group of $D_{\mathbf{Q}_q}$, into $\mathrm{PGL}(2, \mathbf{Q}_q)$, and thus also a map of Γ into $\mathrm{PGL}(2, \mathbf{Q}_q)$; both are denoted by θ_q.

Lemma. *The closure of* $(\theta_p \times \theta_q)(\Gamma)$ *in* $\mathrm{PGL}(2, \mathbf{Q}_p) \times \mathrm{PGL}(2, \mathbf{Q}_q)$ *is all of* $\mathrm{PSL}(2, \mathbf{Z}_p) \times \mathrm{PSL}(2, \mathbf{Z}_q)$.

Proof. The proof follows immediately from the strong approximation theorem of M. Kneser ([16, page 81]), and the fact that $\mathrm{PSL}(2, \mathbf{Z}_p)$ is open in $\mathrm{PSL}(2, \mathbf{Q}_p)$. □

Corollary. *The closure of* $\theta_q(\Gamma(p))$ *is all of* $\mathrm{PSL}(2, \mathbf{Z}_q)$.

Proof. Since $\Gamma(p)$ is of finite index in Γ, $\overline{\theta_p(\Gamma(p))} \times \mathrm{PSL}(2, \mathbf{Z}_q)$ is open and thus by the lemma $(\theta_p \times \theta_q)(\Gamma)$ is dense there. However,

$$\theta_p^{-1}(\overline{\theta_p(\Gamma(p))} \cap \theta_p(\Gamma)) \subset \Gamma(p)$$

and thus $\theta_q(\Gamma(p))$ is dense in $\mathrm{PSL}(2, \mathbf{Z}_q)$. □

Recall that $\delta/\sqrt{q}$ together with $\Gamma(p)$ generates a finite extension of $\Gamma(p)$. Modulo scalars, the same is true for δ, and thus since the map θ_q incorporates the canonical projection of $\mathrm{GL}(2, \mathbf{Q}_p) \to \mathrm{PGL}(2, \mathbf{Q}_p)$ we have that the group $\{\theta_q(\Gamma(p)), \theta_q(\delta)\}$ generated by $\theta_q(\Gamma(p))$ and $\theta_q(\delta)$ is a finite extension of $\theta_q(\Gamma(p))$. It follows that the group

$$K = \overline{\{\theta_q(\Gamma(p), \theta_q(\delta)\}}$$

is a finite extension of $\mathrm{PSL}(2, \mathbf{Z}_q)$ and thus a compact subgroup of $\mathrm{PGL}(2, \mathbf{Q}_q)$ that contains both $\mathrm{PSL}(2, \mathbf{Z}_q)$ and $\theta_q(\delta)$.

Let X be the tree of equivalence classes of lattices in the 2-dimensional vector space $V = \mathbf{Q}_q^2$ over $\mathbf{Q}_q$ (L and L' being equivalent if $L' = tL$ for some $t \in \mathbf{Q}_q^*$). $\mathrm{PGL}(2, \mathbf{Q}_q)$ acts on X and by prop. 2, chapter II of [14], K being a finite extension of $\mathrm{PSL}(2, \mathbf{Z}_q)$, fixes a vertex Λ_0 of X. By the corollary of proposition 1 of chapter II in [14] we have

$$d(\Lambda, s\Lambda) \equiv v(\det(s)) \pmod 2$$

where d denotes the distance function on X, $\Lambda \in X$, $s \in \mathrm{GL}(2, \mathbf{Q}_q)$ and v is the valuation on $\mathbf{Q}_q$. In this formula one can take s to be an element of $\mathrm{PGL}(2, \mathbf{Q}_q)$ where the determinant is taken for some representative of s in $\mathrm{GL}(2, \mathbf{Q}_q)$. In particular, since $\det \delta = q$ we get

$$d(\Lambda, \theta_q(\delta)\Lambda) \equiv v(\det(\delta)) = v(q) \equiv 1 \pmod 2.$$

On the other hand, since $\theta_q(\theta) \subset K$ we have $\theta_q(\delta)\Lambda_0 = \Lambda_0$. This contradiction completes the proof of Theorem 1. □

§3. Remarks

(a) If one is interested in just a single pair of horocycle flows on compact surfaces that have no common factor but are not disjoint, a more geometric example is available for which we are indebted to H. Farkas and L. Greenberg. The two groups Γ_1, Γ_2 in question are the so-called triangle groups $\Gamma_1 = T(2, 3, 9)$, $\Gamma_2 = T(2, 3, 18)$. On the one hand, these groups are not isomorphic and are known to be maximal in the class of Fuchsian groups, and so by Theorems A–C they have no common factor. On the other hand, from the general inclusions $T(m, m, n) \subset T(2, m, 2n)$ with index 2, $T(3, 3, 9) \subset \Gamma_2$ with index 2, while

$T(3,n,3n)\subset T(2,3,3n)$ with index 4 implies that $T(3,3,9)\subset\Gamma_1$ with index 4, hence there is a common finite extension of the horocycle flows G/Γ_1 and G/Γ_2, namely $G/\Gamma(3,3,9)$.

The assertions used above concerning the maximality of the triangle groups in question are contained in [7] and [15]. The inclusion can, of course, be easily checked directly.

(b) Using Theorems B and C alone, one sees that the examples that we constructed give a negative answer to the measure theoretic version of Furstenberg's question.

Our example differs from the one described in [13] in that the joining in our case is a finite extension of both transformations, whereas in their case the joining is a two point extension of one of the transformations but a continuous extension of the other.

(c) Let $\mathscr{X}=(X,\mathscr{B},\mu,T)$ be an ergodic measure preserving system where X is compact metric and T a homeomorphism of X. Let $\mathscr{P}(X)$ be the space of probability measures on X with the weak $*$ topology and $\mathscr{G}$ the corresponding Borel field. We use the same letter T to denote the homeomorphism induced by T on $\mathscr{P}(X)$. If λ is a probability measure on $\mathscr{P}(X)$ we say that $\mathscr{Y}=(\mathscr{P}(X),\mathscr{G},\lambda,T)$ is a *quasi-factor* (q.f.) of $\mathscr{X}$ if λ is T invariant and for each $f\in C(X)$

$$\int_X f(x)d\mu=\int\int_{\mathscr{P}(X)X} f(x)d\nu(x)d\lambda(\nu).$$

It can be shown that as a measure theoretical object a q.f. is an invariant of the original measure theoretical process. For more details the reader is referred to [6].

Given Γ_p, Γ_q $(p,q\in P)$ as in §2 we have a natural homeomorphism of $G/\Gamma_p\cap\Gamma_q$ onto a subspace of $(G/\Gamma_p)\times(G/\Gamma_q)$, namely $g(\Gamma_p\cap\Gamma_q)\rightarrow(g\Gamma_p,g\Gamma_q)$. Let μ, λ and θ denote the invariant measures on $X=G/\Gamma_p$, $y=G/\Gamma_q$ and $G/\Gamma_p\cap\Gamma_q$, respectively.

Disintegrating θ over λ we have

$$\theta=\int_{G/\Gamma_q}\nu_y\times\delta_y d\lambda(y).$$

The map $y\rightarrow\nu_y$ of Y into $\mathscr{P}(X)$ sends λ onto a measure $\tilde{\lambda}$ on $\mathscr{P}(X)$ which defines a q.f. of (X,μ,h_1). Thus for any $q\in P$ there is a q.f. of $(G/\Gamma_p,\mu,h_1)$ which is a factor of $(G/\Gamma_q,\lambda_q,h_1)$. This yields a countable family of non-isomorphic q.f. of (X,μ,h_1).

There exists an n such that the q.f. $(\mathcal{P}(X), G, \tilde{\lambda}, h_1)$ is isomorphic to an ergodic process on $X^{\hat{n}}$, the nth symmetric product of X.

Let $\tilde{\tilde{\lambda}}$ be the unique permutation invariant lift of $\tilde{\lambda}$ to X^n and let λ_0 be an ergodic component of $\tilde{\tilde{\lambda}}$. It is easy to check that the projection of λ_0 on each X component is μ. If we consider any projection of λ_0 onto an $X \times X$ component we see that this projection can be neither $\mu \times \mu$ nor $\int \delta_x \times \delta_{h_t x} d\mu(x)$, for some $t \in \mathbf{R}$. The former is impossible since λ_0 is a finite extension of each of its X-projections, and the latter will imply that the support of each ν_y contains a pair x, $h_t x$ which, again, one can check is impossible. Thus $(G/\Gamma_p, \mu, h_1)$ does not have minimal self-joinings in the sense of [13]. For a complete description of the self-joinings of $(G/\Gamma, \mu, h_1)$ see [11], [12].

(d) The methods of [10] can be used to show that every topological factor of a horocycle flow is topologically a horocycle flow. In particular, if Γ is maximal and co-compact $(G/\Gamma, h_t)$ is a real minimal prime flow and $(G/\Gamma, h_1)$ is a prime minimal transformation. Here is a brief sketch of the proof. We suppose that Γ is co-compact and that $\pi : G/\Gamma \to X$ is continuous where X is compact metric and $\pi h_1 = T\pi$ for a continuous transformation $T : X \to X$. An analogous argument can be carried out for the case of the real flow h_t.

(i) A simpler version of the arguments in §§2, 3 of [10] will establish, in this setting (G/Γ compact and π continuous), that there exists a positive constant $c > 0$, such that $x_1 \neq x_2$, $\pi(x_1) = \pi(x_2)$ implies $d(x_1, x_2) \geqq c$. This shows that h_1 is a finite isometric extension of T.

(ii) The unique ergodicity of h_1 shows that the disintegration of the Haar measure on G/Γ with respect to the fibering defined by π is uniformly distributed on the points of the fiber. Thus in case Γ was maximal we are done, since a non-trivial topological factor would give rise to a non-trivial measure theoretic factor which is ruled out by Theorem C.

(iii) An examination of the proof of the main theorem in [10] shows that there is a finite extension of Γ, $\tilde{\Gamma} \supset \Gamma$, such that the canonical projection $\tilde{\pi} : G/\Gamma \to G/\tilde{\Gamma}$ defines a fibering of G/Γ which agrees with the fibering defined by π on a set of full measure. Since the extension is isometric, even a single common fiber would suffice to establish a topological isomorphism between $(G/\tilde{\Gamma}, h_1)$ and (X, T).

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